

AN EVANS-FUNCTION APPROACH TO SPECTRAL STABILITY OF INTERNAL SOLITARY WAVES IN STRATIFIED FLUIDS

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ABSTRACT. Frequently encountered in nature, internal solitary waves in stratified fluids are well-observed and well-studied from the experimental, the theoretical, and the numerical perspective. From the mathematical point of view, these waves are exact solutions of the 2D Euler equations for incompressible, inviscid fluids. Contrasting with a rich theory for their existence and the development of methods for computing these waves, their stability analysis has hardly received attention at a rigorous mathematical level.

This paper proposes a new approach to the investigation of stability of internal solitary waves in a continuously stratified fluidic medium and carries out the following four steps of this approach: (I) to formulate the eigenvalue problem as an infinite-dimensional spatial-dynamical system, (II) to introduce finite-dimensional truncations of the spatial-dynamics description, (III) to demonstrate that each truncation, of any order, permits a well-defined Evans function, (IV) to prove absence of small zeros of the Evans function in the small-amplitude limit. The latter notably implies the low-frequency spectral stability of small-amplitude internal solitary waves to arbitrarily high truncation order.

0. INTRODUCTION

Fluidic media that are stratified according to varying density, as for example lakes, oceans, and atmospheres, typically permit the development and propagation of so-called *internal waves*, which, in contrast to the familiar surface waves, chiefly displace fluid elements far beneath the surface. Internal waves which are close to some quiescent state both far ahead and far astern the wave are called *internal solitary waves* (ISWs). As ISWs provide important mechanisms for mixing and energy transport and thus have direct ecological implications, the fields of oceanography, limnology, and atmosphere science have devoted considerable attention to their observation and description, see [3, 13, 26, 45].

The “channel model” widely used in this context is given by the 2D Euler equations for incompressible fluids posed on a strip of constant finite height and infinite horizontal extent. Mathematical results based on this model broadly show the existence of solitary waves (see below), but the stability of ISWs within the channel model is an open problem. There exist numerous works on comparatively simple model equations, e.g. Korteweg-deVries equation, extended Korteweg-deVries equation, and intermediate long-wave equation, to name just a few, for which the question of stability of solitary waves has been answered comprehensively. These results certainly have implications on the stability properties of ISWs in the full-Euler channel-model setting. But, to the best of our knowledge, no stability analysis has been conducted in this setting as yet.

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The central object of the present paper is a sequence of $4(N+1)$ -dimensional systems, $(E_N)_{N \in \mathbb{N}}$, of ordinary differential equations on the real line that are associated with the spectral stability problem of ISWs in the channel model. We prove two main results on these “truncated eigenvalue problems”, one for essentially arbitrary ISWs, the other for ISWs of small amplitude. The result for arbitrary ISWs (Theorem III in Section 3) establishes what is called consistent splitting on the closed right half $\overline{\mathbb{C}_+}$ of the complex plane, and thus the possibility of properly defining an Evans function, D_N , on $\overline{\mathbb{C}_+}$ for each E_N . The result for small-amplitude ISWs (Theorem IV in Section 4) shows that each E_N has no bounded solutions for small non-negative-real-part values of the spectral parameter, i.e., there do not exist unstable modes of small frequency.

Before obtaining these rigorous results on the truncated problems E_N , we motivate the E_N from the linearization of the full Euler equations about the ISW profile. This is done in two steps. The first step consists in showing that the eigenvalue problem derived from this linearization can be cast in a spatial-dynamics formulation, E , on $(\mathcal{L}^2(0,1))^4$ (Theorem I in Section 2). In the second step, we apply a Galerkin type procedure to E and obtain the ‘truncations’ E_N (Theorem II in Section 2). We emphasize that these ‘derivations’ of E and the E_N are completely formal; no attempts are made in the present paper to give the (certainly ill-posed!) ‘infinite-dimensional dynamical system’ E a rigorous interpretation, to even only formulate spectral stability at its level, or to show that the E_N approximate E in a rigorous sense.¹

It does seem to the author, however, that Theorems III and IV would be very strange coincidences if the E_N did not, despite the formal nature of their deduction in Theorems I and II, capture essential features of the ISW stability problem in the original full-Euler channel-model setting. In particular, we consider Theorem III as a meaningful characterization of stability properties of internal solitary waves of arbitrary amplitude and Theorem IV as significant evidence for the stability of internal solitary waves of small amplitude.

In the following, we recapitulate previous work which has motivated our approach. Kirchgässner proposed what is now, generally, called “spatial dynamics” in his study [32] of an elliptic PDE² posed, as in our context, on a two-dimensional channel. In this approach, the unbounded spatial variable $-\infty < x < +\infty$ is considered as the “time” of a dynamical system living on some function space. Although ill-posed, this spatial-dynamics formulation permits the application of dynamical-systems methods, in particular the centre-manifold reduction, and thus yields new insights into the problem; we refer to [24, 50, 27] for extensive material on that.

The far-reaching spatial-dynamics approach was later also applied in studying the stability of waves. In [25], Haragus and Scheel used such a formulation to prove spectral stability of capillary-gravity surface waves. The idea of our spatial-dynamics approach to stability is close to theirs in principle but differs at prominent places, notably in the lack of a finite-dimensional centre manifold.

When focussing on internal waves of sufficiently small amplitude, it is well known that solitary waves are, to leading order, modelled by solitons of the Korteweg-deVries equation, see [9, 6, 33, 28]. These KdV solitons exhibit a remarkable stability as shown by Benjamin [7], by Bona et al. [12], and by Pego and Weinstein [42, 43]. In [42], Pego and Weinstein developed a unified framework for the stability of solitons for a class of Hamiltonian PDEs by relating conserved quantities of a given soliton to properties of the Evans function associated with it and

¹But these three questions admittedly are topics of ongoing work.

²He did indeed consider a variant of the Dubreil-Jacotin equation that governs ISWs, see Section 1 below.

this work proves, in particular, the spectral stability of KdV solitons that we will exploit in the small-amplitude limit.

Even though this paper passes only formally by the infinite-dimensional spatial-dynamics setting, the large body of work by Latushkin and collaborators towards infinite-dimensional Evans functions (see [22, 37] and references therein) has been a prime motivation, and still is.

We finally remark that the interest in internal solitary waves is also indicated by many publications dealing with algorithms for their quantitative computation (e.g. see [48, 47, 31]). These methods make it possible to investigate properties of internal waves numerically. It is a stimulating question whether numerically observed instabilities of ISWs, such as those reported in [47, 14, 44], can be captured by their spectral properties, i.e., by the emergence of unstable eigenvalues.

Note. This paper presents central results of the author's PhD thesis [34].

1. STATEMENT OF THE RESULTS

In the mathematical modelling of internal waves, it is common practice to consider a two-dimensional channel,

$$\mathfrak{C} = \{(x, y) : x \in \mathbb{R}, 0 < y < 1\},$$

which is entirely filled with a non-homogeneous, inviscid, incompressible fluid, with the density stratification of the fluid at rest being given by a known differentiable function $\bar{\rho}(y)$ satisfying $\bar{\rho}(y) > 0$ and $\bar{\rho}'(y) < 0$ for all $y \in [0, 1]$; such a $\bar{\rho}$ is called a *stable stratification*. A prototypical example is given by the exponential stratification, $\bar{\rho}(y) = e^{-\delta y}$ with some fixed $\delta > 0$, to which we restrict from Thm. II on.

The motion of the fluid is assumed to be governed by the Euler equations³,

$$\begin{aligned} (1.1a) \quad & \rho_t + u\rho_x + v\rho_y = 0, \\ (1.1b) \quad & \rho(u_t + uu_x + vv_y) = -p_x, \\ (1.1c) \quad & \rho(v_t + uv_x + vv_y) = -p_y - g\rho, \end{aligned}$$

complemented by the incompressibility constraint

$$(1.1d) \quad u_x + v_y = 0,$$

with t , x and y denoting the time, horizontal and vertical position, respectively, whereas the sought functions, occasionally collected in the vector U , are given by density $\rho(t, x, y)$, velocity field $(u(t, x, y), v(t, x, y))$, and pressure $p(t, x, y)$; the constant g denotes acceleration due to gravity.

The requirement that the fluid cannot leave the domain \mathfrak{C} is encoded in the boundary conditions

$$(1.2) \quad v(t, x, 0) = 0 \quad \text{and} \quad v(t, x, 1) = 0,$$

the second of which is often referred to as the *rigid lid* condition expressing the fact that in typical applications free-surface displacements are negligible.

In the present setting, the quiescent state $\bar{U}(y) = (\bar{\rho}(y), \bar{u}(y), \bar{v}(y), \bar{p}(y))^T$ with

$$\bar{u}(y) = 0, \quad \bar{v}(y) = 0, \quad \bar{p}(y) = -g \int_0^y \bar{\rho}(\eta) d\eta$$

³In this paper, we do not enter any issues related to the very interesting general solution theory of the Euler equations for heterogeneous incompressible fluids. See [5, 40].

is, in fact, a stationary solution of the Euler equations (1.1) and satisfies (1.2). This permits to precisely define the waves of interest: A function $U^c(\xi, y)$ is called an *internal solitary wave (ISW)* of speed c if

$$U(t, x, y) = U^c(x - ct, y)$$

is a classical solution of (1.1) satisfying (1.2) and if the wave profile $U^c(\xi, y)$ tends to the quiescent state $\bar{U}(y)$ uniformly in y as $|\xi| \rightarrow \infty$.

The inception of rigorous mathematical investigations on internal waves is indicated by the remarkable observation due to Dubreil-Jacotin in [16] and Long in [39] that travelling wave profiles can be found by solving a single nonlinear elliptic equation (for the stream function), the *Dubreil-Jacotin-Long equation*, depending parametrically on the wave speed c ; for a related equation, see [51].

Based on these equations, proofs for the existence of periodic and solitary internal travelling waves have been given via different methods in particular through bifurcation theory for elliptic equations (see [49, 2]), by the use of various variational principles (see [6, 11, 48, 36]), and, for small-amplitude waves, via the spatial-dynamics approach due to Kirchgässner (see [32, 33, 28]).

In order to study the stability of ISWs, we consider the Euler eigenvalue problem. Using the time-exponential perturbation

$$(e^{\kappa t} \rho(\xi, y), e^{\kappa t} u(\xi, y), e^{\kappa t} v(\xi, y), e^{\kappa t} p(\xi, y))^T$$

in the linearization of (1.1) about an ISW solution $U^c(\xi, y)$, the eigenvalue problem reads

$$(1.3a) \quad -\kappa \rho = (u^c - c) \rho_\xi + v^c \rho_y + u \rho_\xi^c + v \rho_y^c,$$

$$(1.3b) \quad -\rho^c \kappa u = \rho^c ((u^c - c) u_\xi + u u_\xi^c + v^c u_y + v u_y^c) \\ + \rho ((u^c - c) u_\xi^c + v^c u_y^c) + p_\xi,$$

$$(1.3c) \quad -\rho^c \kappa v = \rho^c ((u^c - c) v_\xi + u v_\xi^c + v^c v_y + v v_y^c) \\ + \rho ((u^c - c) v_\xi^c + v^c v_y^c) + p_y + g \rho,$$

$$(1.3d) \quad 0 = u_\xi + v_y.$$

A number $\kappa \in \mathbb{C}$ with $\operatorname{Re} \kappa > 0$ is called an *unstable eigenvalue* if this system possesses a bounded solution for the given κ . To show spectral stability of the ISWs, we have to exclude unstable eigenvalues. We are aware that the meaning of *eigenvalue* is vague here as we do not provide a functional-analytic framework; however, there is a clear meaning for the truncated systems which form the core of our approach.

The present work is divided into three parts corresponding with Secs. 2, 3, and 4: (1) establishing, at a formal level, a spatial-dynamics formulation for the eigenvalue problem, and corresponding finite-dimensional truncations (Thms. I and II); (2) showing, rigorously, that the latter are amenable to the Evans function method (Thm. III); (3) proving the absence of unstable modes close to the origin in the small-amplitude limit (Thm. IV).

Now, to explain our results, let $\psi(\xi, y)$ denote the stream function associated with the linearized velocity field,

$$\psi_\xi = -v \text{ and } \psi_y = u.$$

We will consider the underlying space

$$\mathcal{W} = \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1)$$

endowed with the scalar product

$$(1.4) \quad \langle U, V \rangle := \int_0^1 (-\bar{\rho}') \left(\frac{U_1 V_1}{(\bar{\rho}')^2} + U_2 V_2 + U_3 V_3 + U_4 V_4 \right) dy.$$

Since we assume $\bar{\rho}' < 0$ on the closed interval $[0, 1]$, $\langle \cdot, \cdot \rangle$ is obviously equivalent to the standard scalar product and hence $(\mathcal{W}, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Our first (formal!) result will be as follows.

Theorem I (abbreviated statement). *Given an ISW $U^c(\xi, y)$, the associated eigenvalue problem (1.3) can be written as an abstract ordinary differential equation, posed in \mathcal{W} , of the form*

$$(E) \quad W'(\xi) = \mathbb{A}(\xi; \kappa) W(\xi),$$

in which the dependent variable assumes, at “time” ξ , a value

$$(1.5) \quad W(\xi) = (\rho(\xi, \cdot), \psi(\xi, \cdot), \psi_\xi(\xi, \cdot), \psi_{\xi\xi}(\xi, \cdot))^T \in \mathcal{W}$$

and the coefficient \mathbb{A} is of the form

$$(1.6) \quad \mathbb{A}(\xi; \kappa) = \begin{pmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ S_1 & S_2 & S_3 & S_4 \end{pmatrix},$$

where R_1, \dots, S_4 denote appropriate linear operators in $\mathcal{L}^2(0, 1)$.

This theorem is of central importance for our approach as it opens the door for the theory of dynamical systems.

Since the equation (E) represents an abstract ODE on a Hilbert space of infinite dimension, we propose to consider finite-dimensional truncations of (E) in the spirit of [22, 41, 38]. Having found a suitable Hilbert basis \mathfrak{B} of \mathcal{W} , we are able to formulate formal Galerkin-type approximants; this is the content of Thm. II.

Theorem II (abbreviated statement). *There exists a natural sequence of finite-dimensional truncations*

$$(1.7) \quad \hat{W}'_N(\xi) = \hat{\mathbb{A}}_N(\xi; \kappa) \hat{W}_N(\xi), \quad \text{for } N = 0, 1, 2, \dots$$

of (E) such that the operator $\hat{\mathbb{A}}_N(\xi; \kappa)$ has the following matrix representation in the basis \mathfrak{B} :

$${}^N \mathcal{A}(\xi; \kappa) = \begin{pmatrix} \mathcal{A}_0 + \mathcal{B}_{0,0} & \mathcal{B}_{0,1} & \cdots & \mathcal{B}_{0,N} \\ \mathcal{B}_{1,0} & \mathcal{A}_1 + \mathcal{B}_{1,1} & \cdots & \mathcal{B}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{N,0} & \mathcal{B}_{N,1} & \cdots & \mathcal{A}_N + \mathcal{B}_{N,N} \end{pmatrix}$$

where, with $\lambda \equiv \frac{q}{c^2}$,

$$(1.8) \quad \mathcal{A}_M = \mathcal{A}_M(\kappa) = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta\lambda\kappa & -\frac{1}{c}\kappa\lambda_M\delta & \delta(\lambda_M - \lambda) & \frac{\kappa}{c} \end{pmatrix}$$

and $\mathcal{B}_{ML} = \mathcal{B}_{ML}(\xi; \kappa)$ with

$$\mathcal{B}_{ML}(\pm\infty; \kappa) = 0.$$

The representation of system (1.7) in the basis \mathfrak{B} will be denoted by E_N . The treatment of the truncations (E_N) by an Evans function approach is considered next.

The *Evans function* is a tool to detect the point spectrum of differential operators. The main idea is that under certain assumptions eigenvalues of the linearized

operator can be found as the roots of this analytic function. Originally defined for travelling waves in reaction-diffusion equations, the Evans function was later substantially extended to cover viscous conservation laws and dispersive equations as well, see [17, 1, 42, 21, 20] and references therein; we refer to [46] for an extensive introduction.

That the finite-dimensional ODE systems (E_N) are amenable to standard Evans function theory, is accomplished in Thm. III.

Theorem III (abbreviated statement). *Consider a regular ISW of some speed $c > c_0$ and the associated truncated problem (E_N) for an arbitrary $N \in \mathbb{N}$. Then, there exist an open domain $\Omega = \Omega(N, c) \subset \mathbb{C}$ comprising the closed right half-plane $\overline{\mathbb{C}_+}$ and an analytic function $D_N : \Omega \rightarrow \mathbb{C}$ such that (E_N) has a bounded solution for $\kappa \in \mathbb{C}_+$ if and only if $D_N(\kappa) = 0$.*

The guiding idea in the background is that sequences of appropriately scaled ('truncated eigen-') functions \hat{W}_N and Evans functions D_N converge, as $N \rightarrow \infty$, to eigenfunctions \hat{W} and an Evans function D of the original infinite-dimensional problem (E), and \hat{W} yields a solution to (1.3). Important as they are, these issues are not yet considered in the present paper.

Finally, we consider the small-amplitude limit. In that case, we are able to prove the absence of zeros of D_N in a certain neighbourhood of the origin.

Theorem IV. *For all $N \in \mathbb{N}$ and any $R_0 > 0$ there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the Evans function $D_{N,\varepsilon}(\kappa)$ associated with an ISW of amplitude ε^2 satisfies*

$$D_{N,\varepsilon}(0) = 0, \quad D'_{N,\varepsilon}(0) = 0$$

and

$$D_{N,\varepsilon}(\kappa) \neq 0 \text{ for all } \kappa \in \overline{\mathbb{C}_+} \setminus \{0\} \text{ with } |\kappa| < R_0 \varepsilon^3.$$

2. THE SPATIAL-DYNAMICS FORMULATION AND ITS TRUNCATIONS

In the present section, we prove the announced formulation of the eigenvalue problem as a formal spatial-dynamical system. In the whole section, we systematically disregard questions of regularity, domains and ranges of operators, etc. However, we make the following precise assumptions on the profile U^c .

(A1) Differentiability: The profile satisfies $U^c \in C^3(\mathfrak{C})$.

(A2) Exponential decay: There are constants $C_1, C_2 > 0$ such that

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \xi^\alpha \partial y^\beta} (U^c(\xi, y) - \bar{U}(y)) \right| \leq C_1 e^{-C_2 |\xi|}$$

for all $\alpha, \beta \in \{0, 1, 2, 3\}$ with $0 \leq \alpha + \beta \leq 3$.

(A3) Monotonicity: Each level set $\{(\xi, y) : \rho^c(\xi, y) = \varrho\}$, $\varrho \in \text{range } \bar{\rho}$, can be written as the graph $y = Y^e(\xi)$ of some differentiable function $Y^e : \mathbb{R} \rightarrow [0, 1]$.

By a *regular ISW* we mean an ISW satisfying (A1), (A2) and (A3). Note that for a regular ISW $Y^e(\pm\infty) = \bar{\rho}^{-1}(\varrho)$ because of (A2). These assumptions are natural to make since they are satisfied for small-amplitude waves (which follows from the explicit description of such waves, e.g. in [28]) and since the numerical results reported in [48, 35] suggest that they indeed hold far beyond the small-amplitude regime.

We introduce stream functions ψ^c and ψ associated with the velocity field (u^c, v^c) of the travelling wave and the linearized velocity field (u, v) : The incompressibility constraint (1.1d) in the problem's original formulation implies that the vector fields $(-v^c, u^c)$ and $(-v, u)$, defined on the simply-connected domain \mathfrak{C} , possess potentials ψ^c and ψ satisfying the relations

$$\psi_\xi^c = -v^c, \psi_y^c = u^c \quad \text{and} \quad \psi_\xi = -v, \psi_y = u$$

and, in view of (A2) and (1.2), the boundary conditions

$$\psi^c(\xi, 0) = \psi^c(\xi, 1) = 0 \quad \text{and} \quad \psi(\xi, 0) = \psi(\xi, 1) = 0.$$

The use of the stream function is motivated by Benjamin's version of the Euler equations (see [8, p. 34ff.]).⁴

As already noted in the introduction, the underlying function space is the Hilbert space $\mathcal{W} = (\mathcal{L}^2(0, 1))^4$ with the scalar product $\langle \cdot, \cdot \rangle$ from (1.4).

Theorem I. *Given a regular ISW $U^c(\xi, y)$, the associated eigenvalue problem (1.3) can formally be written as the abstract ordinary differential equation*

$$(E) \quad W'(\xi) = \mathbb{A}(\xi; \kappa)W(\xi),$$

in the space $(\mathcal{W}, \langle \cdot, \cdot \rangle)$ where

$$(2.1) \quad W(\xi) = (\rho(\xi, \cdot), \psi(\xi, \cdot), \psi_\xi(\xi, \cdot), \psi_{\xi\xi}(\xi, \cdot))^T \in \mathcal{W}$$

and

$$(2.2) \quad \mathbb{A}(\xi; \kappa) = \begin{pmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ S_1 & S_2 & S_3 & S_4 \end{pmatrix}$$

with

$$R_j = \frac{\tilde{R}_j}{\psi_y^c - c} \quad \text{and} \quad S_k = \frac{\tilde{S}_k}{(\psi_y^c - c)\rho^c}$$

for $j \in \{1, 2, 3\}$ and $k \in \{1, 2, 3, 4\}$, where

$$\tilde{R}_1 = -\kappa + \psi_\xi^c \partial_y,$$

$$\tilde{R}_2 = -\rho_\xi^c \partial_y,$$

$$\tilde{R}_3 = \rho_y^c,$$

$$\begin{aligned} \tilde{S}_1 = & [-(\psi_y^c - c)(\psi_{\xi yy}^c + \psi_{\xi\xi\xi}^c) + \psi_\xi^c(\psi_{yyy}^c + \psi_{\xi\xi y}^c) \\ & - (\psi_y^c - c)^{-1}\{g\kappa + \kappa\psi_\xi^c\psi_{\xi y}^c\} + \kappa\psi_{\xi\xi}^c] \\ & + [-(\psi_y^c - c)\psi_{\xi y}^c + \psi_\xi^c\psi_{yy}^c - \psi_\xi^c\psi_{\xi\xi}^c + (\psi_y^c - c)^{-1}(g\psi_\xi^c + (\psi_\xi^c)^2\psi_{\xi y}^c)] \partial_y, \end{aligned}$$

$$\begin{aligned} \tilde{S}_2 = & [-\rho_y^c\psi_{\xi y}^c - \rho^c\psi_{\xi yy}^c - \rho^c\psi_{\xi\xi\xi}^c - (\psi_y^c - c)^{-1}(g\rho_\xi^c + \rho_\xi^c\psi_\xi^c\psi_{\xi y}^c) - \kappa\rho_y^c] \partial_y \\ & + [\rho_y^c\psi_\xi^c - \kappa\rho^c] \partial_{yy} + [\rho^c\psi_\xi^c] \partial_{yyy}, \end{aligned}$$

$$\begin{aligned} \tilde{S}_3 = & [\rho_y^c\psi_{yy}^c + \rho_\xi^c\psi_{\xi y}^c + \rho^c\psi_{yyy}^c + \rho^c\psi_{\xi\xi y}^c - \rho_y^c\psi_{\xi\xi}^c \\ & + (\psi_y^c - c)^{-1}(g\rho_y^c + \rho_y^c\psi_\xi^c\psi_{\xi y}^c) - \kappa\rho_\xi^c] \\ & + [-\rho_y^c(\psi_y^c - c) + \rho_\xi^c\psi_\xi^c] \partial_y + [-\rho^c(\psi_y^c - c)] \partial_{yy}, \end{aligned}$$

$$\tilde{S}_4 = [-\rho_\xi^c(\psi_y^c - c) - \kappa\rho^c] + [\rho^c\psi_\xi^c] \partial_y.$$

Proof. Using $\psi^c(\xi, y)$ and $\psi(\xi, y)$, system (1.3) assumes the form

$$(2.3a) \quad -\kappa\rho = (\psi_y^c - c)\rho_\xi - \psi_\xi^c\rho_y + \rho_\xi^c\psi_y - \rho_y^c\psi_\xi$$

$$(2.3b) \quad \begin{aligned} -\kappa\rho^c\psi_y = & \rho^c((\psi_y^c - c)\psi_{\xi y} + \psi_{\xi y}^c\psi_y - \psi_\xi^c\psi_{yy} - \psi_{yy}^c\psi_\xi) \\ & + \rho((\psi_y^c - c)\psi_{\xi y}^c - \psi_\xi^c\psi_{yy}^c) + p_\xi, \end{aligned}$$

$$(2.3c) \quad \begin{aligned} \kappa\rho^c\psi_\xi = & \rho^c(-(\psi_y^c - c)\psi_{\xi\xi} - \psi_{\xi\xi}^c\psi_y + \psi_\xi^c\psi_{\xi y} + \psi_{\xi y}^c\psi_\xi) \\ & + \rho(-(\psi_y^c - c)\psi_{\xi\xi}^c + \psi_\xi^c\psi_{\xi y}^c) + p_y + g\rho. \end{aligned}$$

⁴How this formulation underlies the present approach is explained in more detail in the author's PhD thesis [34, Ch. 2].

In terms of the variables

$$W_1 = \rho, \quad W_2 = \psi, \quad W_3 = \psi_\xi, \quad W_4 = \psi_{\xi\xi},$$

Eq. (2.3a) becomes

$$(\psi_y^c - c)W_1' = (-\kappa + \psi_\xi^c \partial_y)W_1 - \rho_\xi^c \partial_y W_2 + \rho_y^c W_3,$$

from which we read off the expressions R_1, R_2, R_3 .

To eliminate the pressure from Eqs. (2.3b), (2.3c) we consider the equation “ $\partial_y(2.3b) - \partial_\xi(2.3c)$ ”. On the left hand side, we obtain

$$\text{LHS} = -\kappa ((\rho^c \psi_\xi)_\xi + (\rho^c \psi_y)_y)$$

and for the right hand side:

$$\begin{aligned} \text{RHS} = & \rho_y^c ((\psi_y^c - c)\psi_{\xi y} + \psi_{\xi y}^c \psi_y - \psi_\xi^c \psi_{yy} - \psi_{yy}^c \psi_\xi) \\ & + \rho_\xi^c ((\psi_y^c - c)\psi_{\xi\xi} + \psi_{\xi\xi}^c \psi_y - \psi_\xi^c \psi_{\xi y} - \psi_{\xi y}^c \psi_\xi) \\ & + \rho^c ((\psi_y^c - c)\psi_{\xi yy} + \psi_{\xi yy}^c \psi_y - \psi_\xi^c \psi_{yyy} - \psi_{yyy}^c \psi_\xi) \\ & + \rho^c ((\psi_y^c - c)\psi_{\xi\xi\xi} + \psi_{\xi\xi\xi}^c \psi_y - \psi_\xi^c \psi_{\xi\xi y} - \psi_{\xi\xi y}^c \psi_\xi) \\ & + \rho_y ((\psi_y^c - c)\psi_{\xi y}^c - \psi_\xi^c \psi_{yy}^c) \\ & + \rho_\xi ((\psi_y^c - c)\psi_{\xi\xi}^c - \psi_\xi^c \psi_{\xi y}^c - g) \\ & + \rho [((\psi_y^c - c)\psi_{\xi yy}^c - \psi_\xi^c \psi_{yyy}^c) + ((\psi_y^c - c)\psi_{\xi\xi\xi}^c - \psi_\xi^c \psi_{\xi\xi y}^c)]. \end{aligned}$$

By replacing $\rho_\xi = W_1' = R_1 W_1 + R_2 W_2 + R_3 W_3$ and solving for $\psi_{\xi\xi\xi}$, we obtain an equation of the form

$$\begin{aligned} \rho^c (\psi_y^c - c) \psi_{\xi\xi\xi} = & \left(\tilde{S}_1^0 + \tilde{S}_1^1 \partial_y \right) W_1 + \left(\tilde{S}_2^1 \partial_y + \tilde{S}_2^2 \partial_y^2 + \tilde{S}_2^3 \partial_y^3 \right) W_2 \\ & + \left(\tilde{S}_3^0 + \tilde{S}_3^1 \partial_y + \tilde{S}_3^2 \partial_y^2 \right) W_3 + \left(\tilde{S}_4^0 + \tilde{S}_4^1 \partial_y \right) W_4 \end{aligned}$$

where the functions \tilde{S}_k^j are given as follows.

$$\begin{aligned} \tilde{S}_1^0 = & -(\psi_y^c - c) (\psi_{\xi yy}^c + \psi_{\xi\xi\xi}^c) + \psi_\xi^c (\psi_{yyy}^c + \psi_{\xi\xi y}^c) \\ & - (\psi_y^c - c)^{-1} \{g\kappa + \kappa \psi_\xi^c \psi_{\xi y}^c\} + \kappa \psi_{\xi\xi}^c, \\ \tilde{S}_1^1 = & -(\psi_y^c - c) \psi_{\xi y}^c + \psi_\xi^c \psi_{yy}^c - \psi_\xi^c \psi_{\xi\xi}^c + (\psi_y^c - c)^{-1} (g\psi_\xi^c + (\psi_\xi^c)^2 \psi_{\xi y}^c), \\ \tilde{S}_2^1 = & -\rho_y^c \psi_{\xi y}^c - \rho^c \psi_{\xi yy}^c - \rho^c \psi_{\xi\xi\xi}^c - (\psi_y^c - c)^{-1} (g\rho_\xi^c + \rho_\xi^c \psi_\xi^c \psi_{\xi y}^c) - \kappa \rho_y^c, \\ \tilde{S}_2^2 = & \rho_y^c \psi_\xi^c - \kappa \rho^c, \\ \tilde{S}_2^3 = & \rho^c \psi_\xi^c, \\ \tilde{S}_3^0 = & \rho_y^c \psi_{yy}^c + \rho_\xi^c \psi_{\xi y}^c + \rho^c \psi_{yyy}^c + \rho^c \psi_{\xi\xi y}^c - \rho_y^c \psi_{\xi\xi}^c \\ & + (\psi_y^c - c)^{-1} (g\rho_y^c + \rho_y^c \psi_\xi^c \psi_{\xi y}^c) - \kappa \rho_\xi^c, \\ \tilde{S}_3^1 = & -\rho_y^c (\psi_y^c - c) + \rho_\xi^c \psi_\xi^c, \\ \tilde{S}_3^2 = & -\rho^c (\psi_y^c - c), \\ \tilde{S}_4^0 = & -\rho_\xi^c (\psi_y^c - c) - \kappa \rho^c, \\ \tilde{S}_4^1 = & \rho^c \psi_\xi^c. \end{aligned}$$

This yields the asserted expressions for the operators

$$\begin{aligned} \tilde{S}_1 &= \tilde{S}_1^0 + \tilde{S}_1^1 \partial_y, \\ \tilde{S}_2 &= \tilde{S}_2^1 \partial_y + \tilde{S}_2^2 \partial_y^2 + \tilde{S}_2^3 \partial_y^3, \\ \tilde{S}_3 &= \tilde{S}_3^0 + \tilde{S}_3^1 \partial_y + \tilde{S}_3^2 \partial_y^2, \\ \tilde{S}_4 &= \tilde{S}_4^0 + \tilde{S}_4^1 \partial_y. \end{aligned}$$

We finally mention that the operators R_i and S_j are not singular. As U^c is assumed to be a regular ISW, hypothesis (A1) ensures that ρ^c and ψ^c possess the differentiability which is required to make sense of the expressions R_i and \tilde{S}_j . It is well-known (see [48, p. 98f.]) that $\rho^c(\xi, y)$ and $\psi^c(\xi, y) - cy$ are related by $\rho^c(\xi, y) = \bar{\rho}(-\frac{1}{c}(\psi^c(\xi, y) - cy))$, hence hypothesis (A3) ensures that the denominators do not vanish: $0 < \bar{\rho}(1) \leq \rho^c(\cdot, \cdot) \leq \bar{\rho}(0)$ and $\partial_y(\psi^c - cy) = -c\bar{\rho}'(-\frac{1}{c}(\psi^c(\xi, y) - cy))^{-1}\partial_y\rho^c(\xi, y) \neq 0$ (due to local solvability). \square

Next, we turn to the derivation of finite-dimensional truncations of (E) by formally applying a Galerkin-type procedure. It is well-known (see [6, p. 250ff.]) that the operator

$$\frac{1}{\bar{\rho}'}T = \frac{1}{\bar{\rho}'}\partial_y(\bar{\rho}\partial_y) : H^2(0, 1) \cap H_0^1(0, 1) \subseteq \mathcal{L}_{-\bar{\rho}'}^2(0, 1) \rightarrow \mathcal{L}_{-\bar{\rho}'}^2(0, 1),$$

with zero boundary conditions, is self-adjoint, positive and uniformly elliptic; its spectrum therefore consists of a sequence of real eigenvalues $0 < \lambda_0 < \lambda_1 < \dots \rightarrow \infty$ and there exists an orthonormal basis for $\mathcal{L}_{-\bar{\rho}'}^2(0, 1)$ of corresponding eigenfunctions $\{\varphi_n\}_{n \in \mathbb{N}}$, which are normalized and mutually orthogonal, i.e.

$$(2.4) \quad \int_0^1 (-\bar{\rho}')\varphi_n\varphi_m dy = \delta_{nm}.$$

It is obvious now that the scalar product (1.4) on the space $\mathcal{W} = (\mathcal{L}^2(0, 1))^4$ is motivated from this scalar product (2.4). What is more, we can find an explicit, particularly suitable Hilbert basis for \mathcal{W} emanating from the set $\{\varphi_n\}_{n \in \mathbb{N}}$ of eigenfunctions of T . For all $N \in \mathbb{N}$, we set

$$U_N^1 = \begin{pmatrix} \bar{\rho}'\varphi_N \\ 0 \\ 0 \\ 0 \end{pmatrix}, U_N^2 = \begin{pmatrix} 0 \\ \varphi_N \\ 0 \\ 0 \end{pmatrix}, U_N^3 = \begin{pmatrix} 0 \\ 0 \\ \varphi_N \\ 0 \end{pmatrix}, U_N^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varphi_N \end{pmatrix}.$$

Then, as a direct consequence of $\{\varphi_n\}_{n \in \mathbb{N}}$ being a Hilbert basis for $\mathcal{L}_{-\bar{\rho}'}^2(0, 1)$, the set

$$\mathfrak{B} := \{U_N^k : k \in \{1, 2, 3, 4\}, N \in \mathbb{N}\}$$

forms a Hilbert basis of the Hilbert space $(\mathcal{W}, \langle \cdot, \cdot \rangle)$.

For the sake of concreteness, we restrict – as it was done too in [36, 10] – to the important special case of an exponential stratification,

$$(2.5) \quad \bar{\rho}(y) = \exp(-\delta y), \quad \text{with fixed } \delta > 0.$$

The rest of this section serves to prove the following theorem.

Theorem II. *With respect to the Hilbert basis \mathfrak{B} , the infinite-dimensional spatial-dynamics formulation (E) has, for any $N \in \mathbb{N}$, a formal finite-dimensional truncation*

$$(E_N) \quad \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix}_\xi = {}^N\mathcal{A}(\xi; \kappa) \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix},$$

with

$${}^N\mathcal{A}(\xi; \kappa) = \begin{pmatrix} \mathcal{A}_0 + \mathcal{B}_{0,0} & \mathcal{B}_{0,1} & \cdots & \mathcal{B}_{0,N} \\ \mathcal{B}_{1,0} & \mathcal{A}_1 + \mathcal{B}_{1,1} & \cdots & \mathcal{B}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{N,0} & \mathcal{B}_{N,1} & \cdots & \mathcal{A}_N + \mathcal{B}_{N,N} \end{pmatrix}$$

where, with $\lambda \equiv \frac{g}{c^2}$,

$$(2.6) \quad \mathcal{A}_M = \mathcal{A}_M(\kappa) = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta\lambda\kappa & -\frac{1}{c}\kappa\lambda_M\delta & \delta(\lambda_M - \lambda) & \frac{\kappa}{c} \end{pmatrix}$$

and $\mathcal{B}_{ML} = \mathcal{B}_{ML}(\xi; \kappa)$ with

$$\mathcal{B}_{ML}(\pm\infty; \kappa) = 0.$$

Proof. We define

$$X_M := \text{span} \{U_M^1, U_M^2, U_M^3, U_M^4\} \text{ and } \mathcal{W}_N := \bigoplus_{0 \leq M \leq N} X_M,$$

and denote by Q_N the orthogonal projection onto the space \mathcal{W}_N . Clearly, $(\mathcal{W}_N)_{N \in \mathbb{N}}$ is an increasing sequence of subspaces $\mathcal{W}_N \subset \mathcal{W}$ of dimensions

$$d_N := 4N + 4 < \infty.$$

From now on, we fix some $N \in \mathbb{N}$. Starting from (E), the announced truncated problem (E_N) is obtained in two steps: In step one, we replace \mathbb{A} by its projected version $Q_N \mathbb{A} Q_N$ and in step two, we write down the representation of the projected problem in the basis \mathfrak{B} .

Step 1: We split the linear operator \mathbb{A} in two parts,

$$(2.7) \quad \mathbb{A}(\xi; \kappa) = \mathbb{A}^\infty(\kappa) + \mathbb{B}(\xi; \kappa),$$

with

$$\mathbb{A}^\infty(\kappa) := \lim_{\xi \rightarrow \pm\infty} \mathbb{A}(\xi; \kappa) = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{\bar{\rho}'}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g\kappa}{c^2\bar{\rho}} & \frac{\kappa}{c\bar{\rho}}T & -\frac{1}{\bar{\rho}}(T - \frac{g}{c^2}\bar{\rho}') & \frac{\kappa}{c} \end{pmatrix}.$$

A direct calculation shows that

$$(2.8) \quad \begin{aligned} \mathbb{A}^\infty(\kappa)U_M^1 &= \frac{\kappa}{c}U_M^1 + \delta\lambda\kappa U_M^4, \\ \mathbb{A}^\infty(\kappa)U_M^2 &= -\delta\lambda_M \frac{\kappa}{c}U_M^4, \\ \mathbb{A}^\infty(\kappa)U_M^3 &= -\frac{1}{c}U_M^1 + U_M^2 + \delta(\lambda - \lambda_M)U_M^4, \\ \mathbb{A}^\infty(\kappa)U_M^4 &= U_M^3 + \frac{\kappa}{c}U_M^4, \end{aligned}$$

thus each X_M with $1 \leq M \leq N$ is invariant under $\mathbb{A}^\infty(\kappa)$; consequently, \mathcal{W}_N is $\mathbb{A}^\infty(\kappa)$ -invariant as well. As announced above, we construct finite-dimensional versions of (E) by substituting $Q_N \mathbb{A} Q_N$ for \mathbb{A} , so instead of (E) we now consider

$$Q_N W'(\xi) = Q_N (\mathbb{A}^\infty(\kappa) + \mathbb{B}(\xi; \kappa)) Q_N W(\xi).$$

Since we know $\mathbb{A}^\infty(\kappa)\mathcal{W}_N \subset \mathcal{W}_N$, we focus on the following ODE

$$(2.9) \quad W'_N(\xi) = \left(\mathbb{A}^\infty(\kappa)|_{\text{Im } Q_N} + Q_N \mathbb{B}(\xi; \kappa)|_{\text{Im } Q_N} \right) W_N(\xi)$$

for $W_N = Q_N W \in \mathcal{W}_N$.

Step 2: In order to derive (E_N) from (2.9), we expand $W_N \in \mathcal{W}_N$ in the basis \mathfrak{B} , i.e.

$$W_N(\xi) = \sum_{M=0}^N \sum_{k=1}^4 w_M^k(\xi) U_M^k,$$

and introduce the notation

$$\mathcal{A}_M^{lk} := \langle \mathbb{A}^\infty U_M^k, U_M^l \rangle, \quad \mathcal{B}_{ML}^{lk} := \langle \mathbb{B} U_L^k, U_M^l \rangle$$

for $M, L \in \{0, 1, \dots, N\}$ and $l, k \in \{1, 2, 3, 4\}$. We emphasize that, due to the splitting (2.7), we have

$$\mathcal{A}_M = \mathcal{A}_M(\kappa) \quad \text{and} \quad \mathcal{B}_{ML} = \mathcal{B}_{ML}(\xi; \kappa),$$

and notably

$$\mathcal{B}_{ML}(\pm\infty; \kappa) = 0.$$

Furthermore, Eq. (2.8) implies that

$$\mathcal{A}_M = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta\lambda\kappa & -\frac{1}{c}\kappa\lambda_M\delta & \delta(\lambda - \lambda_M) & \frac{\kappa}{c} \end{pmatrix}$$

as claimed in (2.6).

In terms of this notation, we obtain

$$(2.10) \quad \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix}_\xi = \underbrace{\begin{pmatrix} \mathcal{A}_0 + \mathcal{B}_{0,0} & \mathcal{B}_{0,1} & \cdots & \mathcal{B}_{0,N} \\ \mathcal{B}_{1,0} & \mathcal{A}_1 + \mathcal{B}_{1,1} & \cdots & \mathcal{B}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{N,0} & \mathcal{B}_{N,1} & \cdots & \mathcal{A}_N + \mathcal{B}_{N,N} \end{pmatrix}}_{= {}^N\mathcal{A}} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix}$$

with $w_n = (w_n^0, w_n^1, w_n^2, w_n^3)^\top$. □

Note that the matrix ${}^N\mathcal{A}(\xi; \kappa)$ decays – due to hypothesis (A2) – exponentially fast to the constant coefficient matrix

$$(2.11) \quad {}^N\mathcal{A}^\infty(\kappa) = \mathcal{A}_0(\kappa) \oplus \cdots \oplus \mathcal{A}_N(\kappa).$$

The spectrum of ${}^N\mathcal{A}^\infty(\kappa)$ will be described in Lm. 4.

3. EVANS FUNCTIONS FOR THE TRUNCATED EIGENVALUE PROBLEMS

In the present section, we construct an Evans function for each of the finite-dimensional truncated versions (E_N) of the problem (E) obtained in the second part of Sec. 2. The central result is the following theorem on the existence of Evans functions for detecting growing modes.

Theorem III. *Given a regular ISW of speed $c > c_0$ and any $N \in \mathbb{N}$, there exists a well-defined, analytic Evans function*

$$D_N : \Omega \rightarrow \mathbb{C}$$

that has the property

$$(E_N) \text{ possesses a bounded solution for } \kappa \quad \text{iff} \quad D_N(\kappa) = 0$$

for all $\kappa \in \mathbb{C}$ with $\text{Re } \kappa > 0$ and satisfies

$$(3.1) \quad D_N(0) = 0 \quad \text{and} \quad D'_N(0) = 0.$$

This theorem is a consequence of the next lemma on the existence of analytic bundles corresponding to stable and unstable spaces. For brevity, we let $\mathcal{G}_k^n(\mathbb{C})$ denote the Grassmannian of all k -dimensional subspaces of \mathbb{C}^n and consider $\mathcal{G}_k^n(\mathbb{C})$ with its standard structure as a compact complex-analytic manifold (see [23, p. 193ff.]).

Lemma 1. *Consider a regular ISW of some speed $c > c_0$ and the associated truncated problem (E_N) for an arbitrary $N \in \mathbb{N}$. Let*

$$d_N^s := N + 1 \text{ and } d_N^u := 3N + 3.$$

Then, there exist an open domain $\Omega = \Omega(N, c) \subset \mathbb{C}$ comprising the closed right half-plane $\overline{\mathbb{C}_+}$ and complex-analytic mappings

$$\mathcal{S}_N : \Omega \rightarrow \mathcal{G}_{d_N^s}^{d_N}(\mathbb{C}),$$

$$\mathcal{U}_N : \Omega \rightarrow \mathcal{G}_{d_N^u}^{d_N}(\mathbb{C}),$$

such that the following characterization holds for any κ in the open right half-plane \mathbb{C}_+ and any solution $w : \mathbb{R} \rightarrow \mathbb{C}^{d_N}$ of (E_N) :

$$w(0) \in \mathcal{S}_N(\kappa) \text{ iff } w(+\infty) = 0,$$

and

$$w(0) \in \mathcal{U}_N(\kappa) \text{ iff } w(-\infty) = 0.$$

For the rest of this section, $'$ always denotes the derivative with respect to κ . The following observation is crucial for the proof of Lm. 1.

Lemma 2. *For any $N \in \mathbb{N}$ and any $\kappa \in \mathbb{C}$ with $\operatorname{Re} \kappa > 0$, the matrix*

$${}^N\mathcal{A}^\infty(\kappa) \in \mathbb{C}^{d_N \times d_N}$$

possesses $d_N^s := N + 1$ eigenvalues with negative real part and $d_N^u := 3N + 3$ eigenvalues with positive real part.

An immediate consequence of Lm. 2 is that, for all $\kappa \in \mathbb{C}_+$, the underlying space $\mathcal{W}_N \cong \mathbb{C}^{d_N}$ splits into a direct sum,

$$(3.2) \quad \mathcal{W}_N = \mathcal{S}_N^\infty(\kappa) \oplus \mathcal{U}_N^\infty(\kappa),$$

where $\mathcal{S}_N^\infty(\kappa)$, resp. $\mathcal{U}_N^\infty(\kappa)$, denotes the span of all generalized eigenvectors associated with eigenvalues of negative, resp. positive, real part of the matrix ${}^N\mathcal{A}^\infty(\kappa)$; Lm. 2 implies

$$\dim \mathcal{S}_N^\infty(\kappa) = d_N^s \quad \text{and} \quad \dim \mathcal{U}_N^\infty(\kappa) = d_N^u.$$

This property, often referred to as *consistent splitting*, is the key requirement for defining an Evans function for the truncated problems.

Proof of Lm. 2. To begin with, we prove that the spectrum of ${}^N\mathcal{A}^\infty(\kappa)$ does not intersect the imaginary axis. To see this, we show that the existence of an imaginary eigenvalue of ${}^N\mathcal{A}^\infty(\kappa)$ implies $\kappa \in i\mathbb{R}$. Since ${}^N\mathcal{A}^\infty(\kappa)$ is block-diagonal (see (2.11)), it suffices to show this separately for each block

$$\mathcal{A}_M(\kappa) = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta\lambda\kappa & -\frac{1}{c}\kappa\lambda_M\delta & \delta(\lambda_M - \lambda) & \frac{\kappa}{c} \end{pmatrix}$$

(recall that $\lambda = \frac{g}{c^2}$, and for λ_M see paragraph below (2.5)). The characteristic polynomial of \mathcal{A}_M is given by

$$(3.3) \quad \chi_M(\mu; \kappa) := \mu^4 - \frac{2\kappa}{c}\mu^3 + \left(\frac{\kappa^2}{c^2} + \delta(\lambda - \lambda_M)\right)\mu^2 + 2\lambda_M\delta\frac{\kappa}{c}\mu - \frac{\kappa^2}{c^2}\lambda_M\delta,$$

and setting $\mu = i\beta$ leads to the equation

$$A\hat{\kappa}^2 + iB\hat{\kappa} - C = 0$$

for $\hat{\kappa} = -\frac{\kappa}{c}$ with

$$A = \beta^2 + \lambda_M\delta > 0, \quad B = 2\beta^3 + 2\lambda_M\delta\beta, \quad C = \beta^4 + \delta(\lambda_M - \lambda)\beta^2.$$

The calculation

$$\begin{aligned} -B^2 + 4AC &= -4\beta^2(\beta^2 + \lambda_M\delta)^2 + 4(\beta^2 + \lambda_M\delta) \cdot \beta^2(\beta^2 + (\lambda_M - \lambda)\delta) \\ &= -4\beta^2(\beta^2 + \lambda_M\delta)\delta\lambda \leq 0 \end{aligned}$$

implies that $\hat{\kappa} \in i\mathbb{R}$, hence $\kappa \in i\mathbb{R}$.

The first part of the proof implies that the dimension of the stable resp. unstable space is the same for all κ with $\operatorname{Re} \kappa > 0$. For determining their exact dimensions, it thus suffices to consider some special choice of κ , and we choose $\kappa \in \mathbb{R}$, $\kappa > 0$ sufficiently large. To handle this precisely, we introduce $t := \mu^{-1}$ and $k := \left(\frac{\kappa}{c}\right)^{-1}$ and, after multiplying (3.3) by $-t^4k^2$, obtain

$$(3.4) \quad \lambda_M\delta t^4 - 2\lambda_M\delta kt^3 - (1 + \delta(\lambda - \lambda_M)k^2)t^2 + 2kt - k^2 = 0.$$

Using the Newton polygon method, it is possible to provide approximate expressions for the roots $t_j = t_j(k)$, $j \in \{1, 2, 3, 4\}$, of (3.4). The underlying idea is to introduce a rescaling $t = k^\gamma T$ with a suitable exponent γ which has to be chosen such that the equation resulting from (3.4) after rescaling and cancelling the highest common power of k retains at least two summands in the leading order. A systematic way to find the appropriate exponents is offered by the Newton polygon method (see [15, Ch. 2.8]). For (3.4), however, this can also be accomplished directly and yields $\gamma \in \{0, 1\}$.

In the first case, $\gamma = 0$, set $k = 0$ in (3.4) to obtain the equation

$$(3.5) \quad \lambda_M\delta t^4 - t^2 = 0$$

which possesses the roots

$$(3.6) \quad t_{1,2}(0) = 0 \quad \text{and} \quad t_{3,4}(0) = \pm \frac{1}{\sqrt{\lambda_M\delta}}.$$

Since the roots $t_{3,4}(0)$ are simple, they persist under perturbations, hence there are two zeros $t_{3,4}(k)$ of (3.4) with approximate expressions

$$(3.7) \quad t_{3,4}(k) = \pm \frac{1}{\sqrt{\lambda_M\delta}} + o(1).$$

As $t_{1,2}(0) = 0$, we cannot infer the sign of $t_{1,2}(k)$.

In the second case, $\gamma = 1$, we plug the ansatz $t = kT$ into (3.4), and cancel the common factor k^2 . Setting $k = 0$ yields

$$-(T - 1)^2 = 0,$$

thus $T_{1,2}(0) = 1$ have positive real part; as $T_{1,2}(k)$ are continuous with respect to k they have positive real part as well. Turning back to the original variables, this means (3.4) has two roots $t_{1,2}(k)$ close to the origin with approximate expressions

$$(3.8) \quad t_{1,2} = k + o(k).$$

Since $k > 0$ is supposed to be small, the formulas (3.7) and (3.8) imply that there are three roots $(t_{1,2,3})$ with positive real part and one root (t_4) with negative real part, as it was claimed. \square

Next, we investigate the zeros of $\chi_M(\mu; \kappa)$ for $\kappa \in i\mathbb{R}$. For this purpose, we introduce $\kappa = iK$ and $\mu = iB$, with $K \in \mathbb{R}$, and consider in the following the real polynomial $p_K(B) := \chi_M(iB; iK)$,

$$p_K(B) = B^4 + 2KB^3 + (K^2 + \delta(\lambda_M - \lambda))B^2 + 2\lambda_M\delta KB + \delta\lambda_M K^2.$$

Lemma 3. *The polynomial $p_K(B)$ has precisely two real roots for any $K \in \mathbb{R}$, which are distinct for $K \neq 0$.*

B	$p_K(B)$	$p'_K(B)$	$p''_K(B)$	$p'''_K(B)$	$p''''_K(B)$	no. of changes
0	+	+	+	+	+	0
$K/2 + \varepsilon$	+	+	\pm	-	+	2
K	-	+	+	-	+	3
B^*	+	-	+	-	+	4

TABLE 1. Number of sign changes at some distinguished points (with any sufficiently small $\varepsilon > 0$ and some sufficiently large $B^* > 0$)

Proof. Step 1: The transformation $(B, K) \mapsto (-B, -K)$ leaves the polynomial invariant. Thus, it suffices to consider $K \geq 0$. For $K = 0$ the polynomial $p_0(B) = B^4 + \delta(\lambda_M - \lambda)B^2$ has precisely two real roots, namely $B_{1,2} = 0$ (since $\lambda_M - \lambda > \lambda_M - \lambda_0 \geq 0$). Thus, it suffices to treat the case $K > 0$ in the rest of the proof.

Step 2: We show that the polynomial has two roots. Since all the coefficients of $p_K(B)$ are positive, any root is negative. Let us consider the signs of $p_K(B)$ and its derivatives at special values of B in order to apply the Fourier-Budan theorem (see [4] for details). According to Table 1, the difference in the number of sign changes between the points B^* and K and between the points K and $K/2 + \varepsilon$ is one in either case. This implies that each of the intervals (K, B^*) and $(K/2 + \varepsilon, K)$ contains precisely one simple zero for any $K > 0$. Since $\varepsilon > 0$ was arbitrary, we can even conclude that $(K/2, K)$ contains precisely one simple zero.

Step 3: In this final step, we exclude further roots of $p_K(B)$. For sufficiently small $K > 0$, the polynomial has no inflection point (since $p''_K(B)$ does not have a real zero), hence $p_K(B)$ has at most two real roots, and we are done in this case. This means we find some $K_0 > 0$ such that $p_{K_0}(B) > 0$ for all $B \in [0, K_0/2]$; K_0 will be used later. For arbitrary $K > 0$, we show that $p_K(B) > 0$ for all $B \in [0, K/2]$. To this end, we consider the derivative of $p_K(B)$, now viewed as a function of the two variables B and K , with respect to K along rays $B = \gamma K$ with $0 < \gamma < \frac{1}{2}$. We find

$$\begin{aligned}
\frac{d}{dK}p_K(\gamma K) &= \gamma \frac{\partial}{\partial B}p_K(\gamma K) + \frac{\partial}{\partial K}p_K(\gamma K) \\
&= \gamma [4(\gamma K)^3 - 6K(\gamma K)^2 + 2(K^2 + \delta(\lambda_M - \lambda))\gamma K - 2\lambda_M\delta K] \\
&\quad + [2((\gamma K)^2 + \delta\lambda_M)(1 - \gamma)K] \\
&= 4\gamma^2(\gamma - 1)^2K^3 + 2\delta(\gamma(\lambda_M - \lambda) + (1 - 2\gamma)\lambda_M)K,
\end{aligned}$$

hence

$$\frac{d}{dK}p_K(\gamma K) > 0$$

for $K > 0$ and $\gamma \in (0, 1/2]$. Consequently, we can state for any fixed $K > 0$ and any B with $0 < \frac{B}{K} \leq \frac{1}{2}$ that

$$p_K(B) = p_K\left(\frac{B}{K}K\right) > p_{K_0}\left(\frac{B}{K}K_0\right) > 0.$$

Lastly, as $p_K(0) = \delta\lambda_M K^2 > 0$, we actually find $p_K(B) > 0$ for all $B \in [0, K/2]$. This means p_K has no zero in the interval $(0, K/2)$. Together with the result from Step 2, we have thus shown that p_K possesses exactly two real roots, which are simple in the case of $K \neq 0$. \square

In the following lemma, we introduce a suitable notation for the eigenvalues of ${}^N\mathcal{A}^\infty(\kappa)$ and merge the results of Lm. 1 and Lm. 2 to obtain a statement on their behaviour for all κ in some open domain containing $\overline{\mathbb{C}_+}$.

Lemma 4. *For any $c > c_0$ and any fixed $N \in \mathbb{N}$ the following holds with some open domain $\Omega = \Omega(N, c) \supset \overline{\mathbb{C}_+}$: The $4N + 4$ eigenvalues of ${}^N\mathcal{A}^\infty(\kappa)$ can be sorted as continuous functions*

$$\mu_n^{s, u_1, u_2, u_3} : \Omega \rightarrow \mathbb{C}, \quad \kappa \mapsto \mu_n^{s, u_1, u_2, u_3}(\kappa) \quad \text{for any } n \in \{0, \dots, N\},$$

such that the relations

$$\operatorname{Re} \mu_n^s(\kappa) < 0 \quad \text{and} \quad \operatorname{Re} \mu_n^{u_3}(\kappa) > 0$$

and

$$\operatorname{sign} \operatorname{Re} \mu_n^{u_1, u_2}(\kappa) = \operatorname{sign} \operatorname{Re} \kappa$$

are true for any $\kappa \in \Omega$.

Proof. The notation of the eigenvalues has been chosen in such a way that Lm. 2 implies the assertion for all $\kappa \in \mathbb{C}_+$.

For any $\kappa \in i\mathbb{R}$, Lm. 3 implies that these eigenvalues satisfy

$$\operatorname{Re} \mu_n^s(\kappa) < 0, \quad \operatorname{Re} \mu_n^{u_3}(\kappa) > 0 \quad \text{and} \quad \operatorname{Re} \mu_n^{u_1, u_2}(\kappa) = 0$$

for any $n \in \{0, \dots, N\}$. As the map $\kappa \mapsto \mu_n^*(\kappa)$ is continuous, we find an open neighbourhood $U(\kappa)$ of κ on which $\operatorname{Re} \mu_n^s < 0$ and $\operatorname{Re} \mu_n^{u_3} > 0$ still hold. The signs of $\operatorname{Re} \mu_n^{u_1, u_2}$ for $\operatorname{Re} \kappa < 0$ follow from the observation that the characteristic polynomial is left unchanged under the mapping $(\mu, \kappa) \mapsto (-\mu, -\kappa)$.

Defining

$$\Omega := \mathbb{C}_+ \cup \bigcup_{\kappa \in i\mathbb{R}} U(\kappa)$$

concludes the proof. \square

Proof of Lm. 1. By Lm. 4 there exists some open set $\Omega \supset \overline{\mathbb{C}_+}$ such that for any $\kappa \in \Omega$

$$\max_{n \in \{0, \dots, N\}} \operatorname{Re} \mu_n^s(\kappa) < \min_{\substack{j \in \{1, 2, 3\}, \\ n \in \{0, \dots, N\}}} \operatorname{Re} \mu_n^{u_j}(\kappa).$$

Consequently, there is a positive spectral gap between the spaces $\mathcal{S}_N^\infty(\kappa)$ and $\mathcal{U}_N^\infty(\kappa)$ which are defined as the span of all (possibly generalized) eigenvectors associated with the spectral sets

$$\{\mu_n^s(\kappa) : n \in \{0, \dots, N\}\},$$

and

$$\{\mu_n^{u_j}(\kappa) : n \in \{0, \dots, N\}, j \in \{1, 2, 3\}\},$$

respectively.

Roughly speaking, this splitting is transported to $\xi = 0$ by the flow of (E_N) . In fact, hypothesis (A2) on the exponential decay ascertains that the construction of stable and unstable spaces due to Alexander, Gardner, Jones (see [1, Sect. 3.B]) also applies here to yield the spaces $\mathcal{S}_N(\kappa)$ and $\mathcal{U}_N(\kappa)$ (these are $\Phi_\pm(\lambda, \tau_0)$ with $\lambda = \kappa$ and $\tau_0 = 0$ in the notation of [1]), which are complex-analytic with respect to κ and which are unique as complex-analytic continuations of the uniquely defined restrictions to \mathbb{C}_+ .⁵ \square

Remark. We note that, since here a spectral gap between the spaces $\mathcal{S}_N(\kappa)$ and $\mathcal{U}_N(\kappa)$ is maintained as κ crosses the imaginary axis, our situation is much simpler than that of the gap lemma (cf. [21]).

⁵The construction of stable and unstable spaces can also be found in [46, 21].

Proof of Thm. III. In order to define an Evans function, we choose analytic bases (the existence of which is guaranteed by construction due to Kato, see [30, Ch. II. §4.2])

$$\{\zeta_1(\kappa), \dots, \zeta_{d_N^s}(\kappa)\} \text{ and } \{\eta_1(\kappa), \dots, \eta_{d_N^u}(\kappa)\}$$

of $\mathcal{S}_N(\kappa)$ and $\mathcal{U}_N(\kappa)$, respectively. Now, we define the Evans function as

$$D_N(\kappa) := \det(\zeta_1(\kappa), \dots, \zeta_{d_N^s}(\kappa), \eta_1(\kappa), \dots, \eta_{d_N^u}(\kappa)).$$

As the chosen basis vectors are analytic with respect to κ , the mapping $D_N : \Omega \rightarrow \mathbb{C}$ is analytic as well.

Finally, we prove property (3.1) in three steps. In the rest of the proof, we write \mathcal{A} instead of ${}^N\mathcal{A}$ and we make c explicit to stress the dependence of ${}^N\mathcal{A}$ on c .

Step 1: According to [46, Sect. 3.3, and Thm. 4.1] the statement in (3.1) is equivalent to the existence of solutions v_1, v_2 satisfying

$$\begin{aligned} v_{1,\xi}(\xi) &= \mathcal{A}(\xi; 0, c)v_1(\xi), \\ v_{2,\xi}(\xi) &= \mathcal{A}(\xi; 0, c)v_2(\xi) + \mathcal{A}^{(1)}(\xi, c)v_1(\xi), \end{aligned}$$

where $\mathcal{A}^{(1)}(\xi, c)$ is uniquely defined by $\mathcal{A}(\xi; \kappa, c) = \mathcal{A}(\xi; 0, c) + \kappa \mathcal{A}^{(1)}(\xi, c)$. It is this statement we are going to prove.

Step 2: The profile $U^c(\xi, y)$ is a stationary solution of the Euler equations in co-moving coordinates and thus solves the system

$$\begin{aligned} (u^c - c)\rho_\xi^c + v^c \rho_y^c &= 0, \\ \rho^c ((u^c - c)u_\xi^c + v^c u_y^c) &= -p_\xi^c, \\ \rho^c ((u^c - c)v_\xi^c + v^c v_y^c) &= -p_y^c - g\rho^c, \\ u_\xi^c + v_y^c &= 0. \end{aligned}$$

Deriving by ξ and by c , respectively, yields that

$$\tilde{V}_1 = \partial_\xi U^c \quad \text{and} \quad \tilde{V}_2 = \partial_c U^c$$

satisfy

$$(*) \mathcal{L}^c \tilde{V}_1 = 0 \quad \text{and} \quad (**) \mathcal{L}^c \tilde{V}_2 = \tilde{V}_1,$$

i. e. \tilde{V}_1 is an eigenvector and \tilde{V}_2 is a generalized eigenvector associated with the eigenvalue $\kappa = 0$ of \mathcal{L}^c , which is the linearized Euler operator (i.e., essentially the right hand side of (1.3)).

Step 3: By Thm. I relation $(*)$ implies the existence of some V_1 with

$$V_{1,\xi} = \mathbb{A}(\xi; 0, c)V_1(\xi).$$

A slight modification of the proof of Thm. I shows that relation $(**)$ implies the existence of some V_2 with

$$V_{2,\xi} = \mathbb{A}(\xi; 0, c)V_2(\xi) + \mathbb{A}^{(1)}(\xi, c)V_1(\xi).$$

The truncation procedure performed in the second part of Sect. 2 yields functions $v_1, v_2 : \mathbb{R} \rightarrow \mathcal{W}_N$, for any $N \in \mathbb{N}$, with

$$\begin{aligned} v_{1,\xi}(\xi) &= \mathcal{A}(\xi; 0, c)v_1(\xi), \\ v_{2,\xi}(\xi) &= \mathcal{A}(\xi; 0, c)v_2(\xi) + \mathcal{A}^{(1)}(\xi, c)v_1(\xi). \end{aligned}$$

We have thus shown that the assertion in Step 1 is true, and this concludes the proof of property (3.1). Hence, the proof of Thm. III is complete. \square

Remark. (i) As there is more than one choice of bases, the Evans function is unique only up to a non-vanishing factor. This non-uniqueness causes no trouble since it does not affect the location of the zeros. (ii) Property (3.1) is a consequence of translational invariance and the presence of a continuum of travelling waves parametrized

by the speed. This is a well-known property of Evans functions associated with a solitary wave, e.g. cf. [42, p. 72ff.] for the corresponding statement for solitons in the generalized Korteweg-deVries equation and other dispersive equations.

4. LOW-FREQUENCY STABILITY OF SMALL-AMPLITUDE ISWS

1. Small-amplitude expressions and stability result. In this section, we apply the Evans function framework established in Sec. 3 to small-amplitude waves that are approximated by Korteweg-deVries solitons. Based on a concrete description of small waves, we derive explicit expressions for the entries of $\mathcal{A}(\xi; \kappa)$. This section's goal is to preclude unstable modes in a neighbourhood of the origin; the precise statement is as follows.

Theorem IV. *For all $N \in \mathbb{N}$ and any $R_0 > 0$ there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the Evans function $D_{N,\varepsilon}(\kappa)$ associated with an ISW of amplitude ε^2 satisfies*

$$D_{N,\varepsilon}(0) = 0, \quad D'_{N,\varepsilon}(0) = 0$$

and

$$D_{N,\varepsilon}(\kappa) \neq 0 \text{ for all } \kappa \in \overline{\mathbb{C}_+} \setminus \{0\} \text{ with } |\kappa| < R_0 \varepsilon^3.$$

Our proof relies on the slow-fast structure of the problem (E_N) , for any $N \in \mathbb{N}$, and an Evans function treatment of the reduced system, in which we recover the eigenvalue problem associated with a KdV soliton!

The starting point is the following remarkable formula for the stream function $\psi^c(\xi, y)$ of a small-amplitude ISW of speed $c = c_0 + \varepsilon^2$ and amplitude roughly ε^2 :

$$(4.1) \quad \psi^c(\xi, y) = a_\varepsilon(\xi) \varphi_0(y) + O(|a_\varepsilon|^2).$$

By virtue of this formula, small ISWs are, to leading order, a product with separated variables; more precisely, the height-independent part $a_\varepsilon(\xi)$ describes the horizontal propagation while $\varphi_0(y)$ is a height-dependent amplification factor. The function φ_0 , as before, denotes the principal eigenfunction of $\frac{1}{\bar{\rho}} \partial_y (\bar{\rho} \partial_y)$ from Sec. 2 (cf. p. 9). The function $a_\varepsilon(\xi)$ is a symmetric soliton solution of the equation

$$a_{\varepsilon,\xi\xi} = -\frac{\varepsilon^2}{s} a_\varepsilon - \frac{r}{s} a_\varepsilon^2 + O(\varepsilon^4),$$

which involves the $\bar{\rho}$ -dependent coefficients r and s defined by

$$s = -\frac{c_0}{2} \frac{\int_0^1 \bar{\rho} \varphi_0^2 dy}{\int_0^1 \bar{\rho} (\varphi_0')^2 dy} < 0 \quad \text{and} \quad r = -\frac{3}{4} \frac{\int_0^1 \bar{\rho} (\varphi_0')^3 dy}{\int_0^1 \bar{\rho} (\varphi_0')^2 dy}.$$

For our exponential stratification $\bar{\rho}(y) = e^{-\delta y}$, these coefficients are (cf. [9, 28]):

$$s = -\frac{c_0}{2\delta\lambda_0} < 0 \quad \text{and} \quad r = -\frac{3\delta\pi^3(e^{\delta/2} + 1)}{2\left(\frac{1}{4}\delta^2 + \pi^2\right)\left(\frac{1}{4}\delta^2 + 9\pi^2\right)} < 0.$$

Finally, the important simple relationship

$$a_\varepsilon(\xi) = \varepsilon^2 A_*(\varepsilon\xi) + O(\varepsilon^4)$$

expresses $a_\varepsilon(\xi)$ in terms of the KdV soliton

$$(4.2) \quad A_*(\Xi) = -\frac{3}{2r} \operatorname{sech}^2\left(\frac{1}{\sqrt{-s}} \Xi\right)$$

satisfying

$$(4.3) \quad A_{*,\Xi\Xi} = -\frac{1}{s} A_* - \frac{r}{s} A_*^2.$$

The appearance of the KdV equation in the context of small-amplitude ISWs permits to extract the essential part of the lengthy expressions for the operators R_k and S_l given in Thm. I. We exemplify this procedure for the expression

$${}^0\mathcal{A}(\xi, \kappa)_{43} = \langle \mathbb{A}(\xi, \kappa) U_0^3, U_0^4 \rangle.$$

Recalling the notation from Thm. I and using Eq. (4.1), we find

$$\begin{aligned} \langle \mathbb{A}(\xi, \kappa) U_0^3, U_0^4 \rangle &= \int_0^1 (-\bar{\rho}'(y)) S_3(\varphi_0) \varphi_0 dy \\ &= \varepsilon^2 \left(\frac{2\lambda_0 \delta}{c_0} - \frac{3\delta}{c_0} \int_0^1 \bar{\rho}(\varphi_0')^3 dy A_*(\varepsilon \xi) \right) + \text{h.o.t.} \\ (4.4) \quad &= \varepsilon^2 \left(-\frac{1}{s} - \frac{2r}{s} A_*(\varepsilon \xi) \right) + \text{h.o.t.} \end{aligned}$$

Proceeding in this way, we finally end up with the following expressions

$$\begin{aligned} \mathcal{A}_0 &= \begin{pmatrix} \frac{1}{c_0} \kappa + O(\kappa \varepsilon^2) & 0 & -\frac{1}{c_0} + O(\varepsilon^2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_0 \delta \kappa + O(\kappa \varepsilon^2) & -\frac{\lambda_0 \delta}{c_0} \kappa + O(\kappa \varepsilon^2) & \frac{2\lambda_0 \delta}{c_0} \varepsilon^2 + O(\varepsilon^4) & \frac{1}{c_0} \kappa + O(\kappa \varepsilon^2) \end{pmatrix}, \\ \mathcal{A}_n &= \begin{pmatrix} \frac{1}{c_0} \kappa + O(\kappa \varepsilon^2) & 0 & -\frac{1}{c_0} + O(\varepsilon^2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_0 \delta \kappa + O(\kappa \varepsilon^2) & -\frac{\lambda_n \delta}{c_0} \kappa + O(\kappa \varepsilon^2) & \delta(\lambda_n - \lambda_0) + O(\varepsilon^2) & \frac{1}{c_0} \kappa + O(\kappa \varepsilon^2) \end{pmatrix} \end{aligned}$$

for $n \geq 1$, and

$$\mathcal{B}_{nm} = \begin{pmatrix} \varepsilon^3 A'_\varepsilon(\xi) G_{nm}^{11} & \varepsilon^3 A'_\varepsilon(\xi) G_{nm}^{12} & \varepsilon^2 A_\varepsilon(\xi) G_{nm}^{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon^3 A'_\varepsilon(\xi) G_{nm}^{41} & \varepsilon^3 A'_\varepsilon(\xi) G_{nm}^{42} & \varepsilon^2 A_\varepsilon(\xi) G_{nm}^{43} & \varepsilon^3 A'_\varepsilon(\xi) G_{nm}^{44} \end{pmatrix}$$

for $n, m \in \mathbb{N}$ after substituting $a_\varepsilon(\xi) = \varepsilon^2 A_\varepsilon(\xi)$ and neglecting higher order terms (i.e. $O(\varepsilon^4 + \kappa \varepsilon^4)$). All of the constants G_{nm}^{ij} , which are independent of κ and ε , can be computed explicitly; in the sequel, however, only the following three will be important:

$$G_{00}^{41} = \frac{2c_0}{3} \frac{r}{s}, \quad G_{00}^{42} = -\frac{4}{3} \frac{r}{s}, \quad G_{00}^{43} = -\frac{2r}{s}.$$

Note that G_{00}^{43} can be directly read off from Eq. (4.4).

In the rest of this section, we shall be occupied with proving that the autonomous first-order system

$$\begin{aligned} (E_{N,\varepsilon}) \quad & w'(\xi) = {}^N\mathcal{A}_{\kappa,\varepsilon}[A_\varepsilon, B_\varepsilon] w(\xi) \\ (4.5) \quad & \begin{pmatrix} A_\varepsilon(\xi) \\ B_\varepsilon(\xi) \end{pmatrix}' = \begin{pmatrix} \varepsilon B_\varepsilon(\xi) \\ \varepsilon \left(-\frac{1}{s} A_\varepsilon - \frac{r}{s} A_\varepsilon^2 \right) + O(\varepsilon^3) \end{pmatrix} \end{aligned}$$

does not have bounded solutions provided that $|\kappa \varepsilon^{-3}|$ be bounded and ε be small enough.

We divide the statement of the theorem in two parts corresponding to the following two propositions which jointly prove the theorem. The first proposition reduces the dimension of the linear part of the original problem from $4N + 4$ to $2N + 4$. The second proposition states absence of unstable modes in the reduced problem.

Proposition 1. *For any N and for sufficiently small ε the system $(E_{N,\varepsilon})$ possesses a centre manifold.*

We denote the reduced problem by $(\hat{E}_{N,\varepsilon})$.

Proposition 2. *For any $R_0, N \in \mathbb{N}$ and $|\kappa| < R_0 \varepsilon^3$ with $\operatorname{Re} \kappa > 0$ the reduced problem $(\widehat{E}_{N,\varepsilon})$ does not possess bounded solutions.*

The two propositions are proved in the subsequent subsections. In the rest of this section, we arbitrarily fix $N \in \mathbb{N}$ and $R_0 > 0$.

2. Proof of Proposition 1: Centre manifold reduction. By introducing $\Lambda := \kappa \varepsilon^{-3}$, we are in the regime

$$0 \leq |\Lambda| \leq R_0.$$

By scaling the dependent variables as

$$\begin{aligned} w_1(\xi) &= W_1(\xi), & w_{4n+1}(\xi) &= \varepsilon W_{4n+1}(\xi), \\ w_2(\xi) &= W_2(\xi), & w_{4n+2}(\xi) &= \varepsilon W_{4n+2}(\xi), \\ w_3(\xi) &= \varepsilon W_3(\xi), & w_{4n+3}(\xi) &= \varepsilon^2 W_{4n+3}(\xi), \\ w_4(\xi) &= \varepsilon^2 W_4(\xi), & w_{4n+4}(\xi) &= \varepsilon^2 W_{4n+4}(\xi), \end{aligned}$$

for all $n \in \{1, \dots, N\}$, the problem $(E_{N,\varepsilon})$ takes, to leading order, the form

$$\begin{aligned} A'_\varepsilon &= \varepsilon B_\varepsilon, \\ B'_\varepsilon &= \varepsilon \left(-\frac{1}{s} A_\varepsilon - \frac{r}{s} A_\varepsilon^2 \right) + O(\varepsilon^3), \\ W'_1 &= O(\varepsilon), \\ W'_2 &= O(\varepsilon), \\ W'_3 &= O(\varepsilon), \\ W'_4 &= O(\varepsilon), \\ W'_{4n+1} &= O(\varepsilon), \\ W'_{4n+2} &= O(\varepsilon), \\ W'_{4n+3} &= W_{4n+4}, \\ W'_{4n+4} &= \delta(\lambda_n - \lambda_0) W_{4n+3} + O(\varepsilon^2), \end{aligned}$$

for all n with $1 \leq n \leq N$. We introduce the two complementary index sets

$$I_h := \{4n+3, 4n+4 : n = 1, \dots, N\} \text{ and } I_c := \{1, \dots, 4N+4\} \setminus I_h$$

such that $j \in I_c$ iff $W'_j = O(\varepsilon)$. From the form of the equations, we easily infer that, for $\varepsilon = 0$, the set

$$\mathcal{M}_0 := \{W_j = 0 : j \in I_h\}$$

is a centre manifold, which is normally hyperbolic since the partial Jacobian matrix of this system with respect to the variables W_{4n+3}, W_{4n+4} for $n = 1, \dots, N$ has a block structure of the form

$$(4.6) \quad \begin{pmatrix} Y_1 & * & \cdots & * \\ 0 & Y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & Y_N \end{pmatrix}$$

where each “*” denotes a submatrix which is irrelevant in the sequel and Y_n is given by

$$Y_n = \begin{pmatrix} 0 & 1 \\ \delta(\lambda_n - \lambda_0) & 0 \end{pmatrix},$$

thus all the eigenvalues, given by $\pm \sqrt{\delta(\lambda_n - \lambda_0)}$ for $n = 1, \dots, N$, are real and different from zero.

By virtue of Fenichel's theorem on the persistence of normally hyperbolic invariant manifolds (see [18, 19, 29]), we conclude that (a) an invariant manifold \mathcal{M}_ε exists for all sufficiently small ε , say $0 < \varepsilon < \varepsilon_1$, (b) \mathcal{M}_ε is a graph over \mathcal{M}_0 and (c) $W_j = O(\varepsilon)$ for $j \in I_h$.

In this way, we obtain a reduced system for the variables $\{W_j : j \in I_c\}$. After changing to the slow scale

$$\Xi := \varepsilon \xi,$$

setting

$$A_\varepsilon(\xi) = \tilde{A}_\varepsilon(\Xi), \quad B_\varepsilon(\xi) = \tilde{B}_\varepsilon(\Xi)$$

and

$$W_1(\xi) = \frac{1}{2}\hat{W}_1(\Xi) - \frac{1}{c_0}\hat{W}_2(\Xi), \quad W_2(\xi) = \frac{c_0}{2}\hat{W}_1(\Xi) + \hat{W}_2(\Xi)$$

as well as

$$W_j(\xi) = \hat{W}_j(\Xi) \quad \text{for all } j \in I_c \setminus \{1, 2\},$$

we find that, omitting the hats, the reduced system is of the form

$$\begin{aligned} \dot{W}_1 &= O(\varepsilon^2), \\ \dot{W}_2 &= W_3 + O(\varepsilon^2), \\ \dot{W}_3 &= W_4, \\ \dot{W}_4 &= \Gamma_1 W_1 + \Gamma_2 W_2 + \Gamma_3 W_3 + O(\varepsilon), \\ \dot{W}_{4n+1} &= O(\varepsilon), \\ \dot{W}_{4n+2} &= O(\varepsilon), \end{aligned} \quad (\hat{\mathbf{E}}_{N,\varepsilon})$$

with

$$\begin{aligned} \Gamma_1(\Xi) &= \frac{1}{2}\tilde{\Gamma}_1 + \frac{c_0}{2}\tilde{\Gamma}_2 = -\frac{1}{2}\dot{A}_*(\Xi) \int_0^1 \bar{\rho} \varphi_0'^3 dy = -\frac{c_0 r}{3s} \dot{A}_*(\Xi), \\ \Gamma_2(\Xi) &= -\frac{1}{c_0}\tilde{\Gamma}_1 + \tilde{\Gamma}_2 = \frac{\Lambda}{s} - \frac{2r}{s} \dot{A}_*(\Xi), \\ \Gamma_3(\Xi) &= -\frac{1}{s} - \frac{2r}{s} \dot{A}_*(\Xi). \end{aligned}$$

In this system, the KdV eigenvalue problem becomes visible as follows: For $\varepsilon = 0$ we obtain the system

$$\begin{aligned} \dot{W}_1 &= 0, \\ \dot{W}_2 &= W_3, \\ \dot{W}_3 &= W_4, \\ \dot{W}_4 &= \Gamma_1 W_1 + \Gamma_2 W_2 + \Gamma_3 W_3, \\ \dot{W}_{4n+1} &= 0, \\ \dot{W}_{4n+2} &= 0; \end{aligned} \quad (\hat{\mathbf{E}}_{N,0})$$

the set $\{W_1 = 0\}$ is invariant for this flow and on this set the only non-trivial equations are those for W_2, W_3, W_4 :

$$(\mathbf{E}_{\text{KdV}}) \quad \begin{pmatrix} W_2 \\ W_3 \\ W_4 \end{pmatrix}_\Xi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\Lambda}{s} - \frac{2r}{s} \dot{A}_*(\Xi) & -\frac{1}{s} - \frac{2r}{s} \dot{A}_*(\Xi) & 0 \end{pmatrix} \begin{pmatrix} W_2 \\ W_3 \\ W_4 \end{pmatrix}.$$

This is the eigenvalue problem of the KdV equation associated with $A_*(\Xi)$!

3. Proof of Prop. 2: Absence of growing modes in the reduced system $(\widehat{E}_{N,\varepsilon})$. We will use an Evans function argument in order to prove the statement. Therefore, we will first show that the system $(\widehat{E}_{N,\varepsilon})$ is contained in the class of eigenvalue problems treated by Pego and Weinstein in [42]. For this purpose, we have to check their hypotheses (H1-H4) on the smoothness of the matrix ${}^N\mathcal{A}$, limits at infinity, simplicity of the lowest eigenvalue, and integrability of the deviator. As it is easy to see that (H1), (H2) and (H4) are satisfied, we concentrate on (H3), which states that there is a domain in \mathbb{C} such that the asymptotic matrix has a unique simple eigenvalue of smallest real part. To be more precise, let $\chi_\varepsilon(\mu; \Lambda)$ denote the characteristic polynomial of the asymptotic matrix associated with system $(\widehat{E}_{N,\varepsilon})$ for $\varepsilon > 0$; we have the following lemma.

Lemma 5. *There exists some $\varepsilon_2 > 0$ such that $\chi_\varepsilon(\mu; \Lambda)$ has a unique simple eigenvalue of smallest real part for all $0 \leq |\Lambda| \leq R_0$ and for all $0 \leq \varepsilon \leq \varepsilon_2$.*

Proof. Let $\chi_0(\mu; \Lambda)$ and $\chi_{\text{KdV}}(\mu; \Lambda)$ denote the characteristic polynomials of the asymptotic matrices associated with the systems $(\widehat{E}_{N,0})$ and (E_{KdV}) , respectively. By inspection of these matrices, one finds

$$\chi_0(\mu; \Lambda) = (-\mu)^{2N+1} \chi_{\text{KdV}}(\mu; \Lambda).$$

The polynomial $\chi_{\text{KdV}}(\mu; \Lambda) = \mu^3 + \frac{1}{s}\mu - \frac{1}{s}\Lambda$ has the following property (see [42, p. 72]): There exists some $\nu > 0$ such that for all Λ with $\text{Re } \Lambda \geq -\nu$ there is a unique simple root $\mu_{\text{KdV}}(\Lambda)$ with smallest real part (which is negative). Consequently, the polynomial $\chi_0(\mu; \Lambda)$ has this property as well, and we have $\mu_0(\Lambda) = \mu_{\text{KdV}}(\Lambda)$.

In order to show that $\chi_\varepsilon(\mu; \Lambda)$ also possesses this property for sufficiently small ε , we invoke the implicit function theorem. Let us define the compact set

$$K := \{\text{Re } \Lambda \geq -\nu\} \cap \{|\Lambda| \leq R_0\}.$$

The equation

$$(4.7) \quad 0 = \chi(\mu; \varepsilon, \Lambda) := \chi_\varepsilon(\mu; \Lambda)$$

has the solution $\mu_0(\Lambda)$ for $\varepsilon = 0$, i.e. $\chi(\mu_0(\Lambda); 0, \Lambda) = 0$. Since $\mu_0(\Lambda)$ is a simple root of $\chi_0(\mu; \Lambda)$, we know

$$\frac{\partial}{\partial \mu} \chi(\mu_0(\Lambda); 0, \Lambda) \neq 0,$$

thus the implicit function theorem implies that Eq. (4.7) can be solved for μ in a neighbourhood of $(\varepsilon, \Lambda) = (0, \Lambda_0)$ for any $\Lambda_0 \in K$. What is more, for any $\Lambda_0 \in K$ there exist some $\tilde{\varepsilon}_2 > 0$ and a smooth function

$$\tilde{\mu} : (-\tilde{\varepsilon}_2, \tilde{\varepsilon}_2) \times \{\Lambda : |\Lambda - \Lambda_0| < \tilde{\varepsilon}_2\} \rightarrow \mathbb{C}$$

with $\chi(\tilde{\mu}(\varepsilon, \Lambda); \varepsilon, \Lambda) = 0$. In this way, we obtain an open cover of K and its compactness allows to pass to a finite subcover. Hence, we find some $\varepsilon_2 > 0$ and some function

$$\mu : [0, \varepsilon_2) \times K \rightarrow \mathbb{C}$$

such that $\mu_\varepsilon(\Lambda) := \mu(\varepsilon, \Lambda)$ is the unique simple zero of smallest real part of $\chi_\varepsilon(\mu; \Lambda)$ for all $0 < \varepsilon < \varepsilon_2$ and all $\Lambda \in K$. \square

We have thus shown that hypothesis (H3) holds on the domain

$$\Omega := \{\Lambda \in \mathbb{C} : \text{Re } \Lambda > -\nu\} \cap \{|\Lambda| < R_0\}$$

and, hence, we may treat the system $(\widehat{E}_{N,\varepsilon})$ by applying the theory due to Pego and Weinstein. Before doing so, we recapitulate some of their notation.

Recall that for a linear differential equation

$$\frac{dy}{dx} = \mathcal{A}(x)y,$$

where $y(x)$ is a column vector, the adjoint system is given by

$$\frac{dz}{dx} = -z\mathcal{A}(x),$$

where $z(x)$ is a row vector. In the following, we denote by (E_{KdV}^*) , $(\widehat{E}_{N,0}^*)$ and $(\widehat{E}_{N,\varepsilon}^*)$ the adjoint systems of (E_{KdV}) , $(\widehat{E}_{N,0})$ and $(\widehat{E}_{N,\varepsilon})$, respectively.

Let $Z_{\text{KdV}}^+(\Lambda)$, $Z_0^+(\Lambda)$, $Z_\varepsilon^+(\Lambda)$ and $Y_{\text{KdV}}^-(\Lambda)$, $Y_0^-(\Lambda)$, $Y_\varepsilon^-(\Lambda)$ denote associated right, resp. left, eigenvectors of the asymptotic matrices normalized in such a way that $Y^- \cdot Z^+ = 1$ holds. Then, we obtain the following lemma which states the existence of special functions spanning the stable space and the dual of the unstable space, respectively, directly by applying [42, Prop. 1.2] to each of the systems (E_{KdV}) , $(\widehat{E}_{N,0})$ and $(\widehat{E}_{N,\varepsilon})$.

Lemma 6. [42, p. 56, Prop. 1.2] (i) For $0 < \varepsilon < \varepsilon_2$ there are differentiable functions

$$\zeta_{\text{KdV}}^+(\xi; \Lambda), \quad \zeta_0^+(\xi; \Lambda), \quad \zeta_\varepsilon^+(\xi; \Lambda),$$

analytic with respect to $\Lambda \in \Omega$, with the following properties:

$\zeta_{\text{KdV}}^+(\xi; \Lambda)$ solves (E_{KdV}) and satisfies $e^{\mu_{\text{KdV}}(\Lambda)\xi} \zeta_{\text{KdV}}^+(\xi; \Lambda) \rightarrow Z_{\text{KdV}}^+(\Lambda)$ as $\xi \rightarrow \infty$,

$\zeta_0^+(\xi; \Lambda)$ solves $(\widehat{E}_{N,0})$ and satisfies $e^{\mu_0(\Lambda)\xi} \zeta_0^+(\xi; \Lambda) \rightarrow Z_0^+(\Lambda)$ as $\xi \rightarrow \infty$,

$\zeta_\varepsilon^+(\xi; \Lambda)$ solves $(\widehat{E}_{N,\varepsilon})$ and satisfies $e^{\mu_\varepsilon(\Lambda)\xi} \zeta_\varepsilon^+(\xi; \Lambda) \rightarrow Z_\varepsilon^+(\Lambda)$ as $\xi \rightarrow \infty$.

These conditions characterize the functions uniquely up to a constant factor.

(ii) For $0 < \varepsilon < \varepsilon_2$ there are differentiable functions

$$\eta_{\text{KdV}}^-(\xi; \Lambda), \quad \eta_0^-(\xi; \Lambda), \quad \eta_\varepsilon^-(\xi; \Lambda),$$

analytic with respect to Λ , with the following properties:

$\eta_{\text{KdV}}^-(\xi; \Lambda)$ solves (E_{KdV}^*) and satisfies $e^{\mu_{\text{KdV}}(\Lambda)\xi} \eta_{\text{KdV}}^-(\xi; \Lambda) \rightarrow Y_{\text{KdV}}^-(\Lambda)$ as $\xi \rightarrow -\infty$,

$\eta_0^-(\xi; \Lambda)$ solves $(\widehat{E}_{N,0}^*)$ and satisfies $e^{\mu_0(\Lambda)\xi} \eta_0^-(\xi; \Lambda) \rightarrow Y_0^-(\Lambda)$ as $\xi \rightarrow -\infty$,

$\eta_\varepsilon^-(\xi; \Lambda)$ solves $(\widehat{E}_{N,\varepsilon}^*)$ and satisfies $e^{\mu_\varepsilon(\Lambda)\xi} \eta_\varepsilon^-(\xi; \Lambda) \rightarrow Y_\varepsilon^-(\Lambda)$ as $\xi \rightarrow -\infty$.

These conditions characterize the functions uniquely up to a constant factor.

With these functions at hand, we can define the Evans functions

$$D_{\text{KdV}}(\Lambda) := \eta_{\text{KdV}}^-(\xi; \Lambda) \cdot \zeta_{\text{KdV}}^+(\xi; \Lambda),$$

$$\hat{D}_0(\Lambda) := \eta_0^-(\xi; \Lambda) \cdot \zeta_0^+(\xi; \Lambda),$$

$$\hat{D}_\varepsilon(\Lambda) := \eta_\varepsilon^-(\xi; \Lambda) \cdot \zeta_\varepsilon^+(\xi; \Lambda)$$

for the systems (E_{KdV}) , $(\widehat{E}_{N,0})$ and $(\widehat{E}_{N,\varepsilon})$ in the vein of Pego and Weinstein. For later use, we recall their result on D_{KdV} .

Lemma 7. The Evans function $D_{\text{KdV}} : \Omega_{\text{KdV}} \rightarrow \mathbb{C}$ is analytic on the domain $\Omega_{\text{KdV}} = \{\Lambda \in \mathbb{C} : \text{Re } \Lambda > -\nu\}$, with some $\nu > 0$, and has the following properties:

- (i) $D_{\text{KdV}}(0) = D'_{\text{KdV}}(0) = 0$, $D''_{\text{KdV}}(0) \neq 0$, and
- (ii) $D_{\text{KdV}}(\Lambda) \neq 0$ for all $\Lambda \neq 0$ with $\text{Re } \Lambda \geq 0$.

So far, we have gathered all the ingredients necessary to give the proof of Prop. 2.

Proof of Prop. 2. Step 1: For $\varepsilon = 0$, we find special solutions $\tilde{\zeta}_0^+$, $\tilde{\eta}_0^-$ to $(\widehat{E}_{N,0})$ and its adjoint system, namely

$$\tilde{\zeta}_0^+ = (0, \zeta_{\text{KdV}}^+, 0, \dots, 0)^\top,$$

$$\tilde{\eta}_0^- = (*, \eta_{\text{KdV}}^-, 0, \dots, 0),$$

(with $*$ appropriately chosen) exhibiting the correct decay rate $\mu_0 = \mu_{\text{KdV}}$ for $\xi \rightarrow \pm\infty$, respectively. Therefore, Lm. 6 implies that there are complex constants $\gamma_1, \gamma_2 \in \mathbb{C}$ such that

$$\zeta_0^+ = \gamma_1 \tilde{\zeta}_0^+ \quad \text{and} \quad \eta_0^- = \gamma_2 \tilde{\eta}_0^-,$$

hence

$$\hat{D}_0(\Lambda) = \eta_0^- \cdot \zeta_0^+ = \gamma_1 \gamma_2 \tilde{\eta}_0^- \cdot \tilde{\zeta}_0^+ = \gamma \eta_{\text{KdV}}^- \cdot \zeta_{\text{KdV}}^+ = \gamma D_{\text{KdV}}(\Lambda)$$

with $\gamma := \gamma_1 \gamma_2$ being constant. Thus, Lm. 7 implies that $\hat{D}_0(\Lambda)$ does not vanish in Ω except for $\Lambda = 0$ where a double zero is present.

Step 2: By Lm. 6 the functions $\zeta_\varepsilon^+, \eta_\varepsilon^-$, hence $\hat{D}_\varepsilon(\Lambda)$, also exist for $0 \leq \varepsilon < \varepsilon_2$.

The Evans function $\hat{D}_\varepsilon(\Lambda)$ is analytic in Λ (and continuous with respect to ε), thus we can compare the numbers of zeros of \hat{D}_ε and \hat{D}_0 inside a given domain by invoking Rouché's theorem. First, we consider a small open ball U_0 centred at the origin. By choosing ε sufficiently small, say $0 \leq \varepsilon < \varepsilon_3$ we ensure that $|\hat{D}_\varepsilon(\Lambda) - \hat{D}_0(\Lambda)| < |\hat{D}_0(\Lambda)|$ holds on the boundary ∂U_0 ; this is possible since \hat{D}_ε is continuous in ε , coincides with \hat{D}_0 for $\varepsilon = 0$, and $\hat{D}_0(\Lambda)$ does not vanish on ∂U_0 . Therefore, the number of zeros of \hat{D}_ε in U_0 equals the number of zeros of \hat{D}_0 in U_0 , which is two by the previous step.

Second, for any open ball $U \subset \overline{\Omega} \setminus U_0$, we similarly find that \hat{D}_ε does not vanish on U provided ε is sufficiently small. Since $\overline{\Omega} \setminus U_0$ is compact, we may pass to a finite subcover to conclude that there exists some $\varepsilon_4 > 0$ such that \hat{D}_ε does not vanish on $\overline{\Omega} \setminus U_0$ for all $0 \leq \varepsilon < \varepsilon_4$.

Step 3: On the other hand, \hat{D}_ε has a double zero in $\Lambda = 0$ due to the generalized eigenvectors associated with horizontal shifts and changes in speed, see Thm. III. As we have shown that there are at most two zeros, we see that $\hat{D}_\varepsilon(\Lambda) \neq 0$ for all $\Lambda \in \Omega \setminus \{0\}$ with $\text{Re } \Lambda \geq 0$ and for all $0 \leq \varepsilon < \varepsilon_0 := \min\{\varepsilon_k : k = 1, \dots, 4\}$; this concludes the proof. \square

5. PERSPECTIVE

In order to illustrate the wider perspective of our approach, we finally give the following conjecture.

Conjecture. *Consider a regular ISW $U^c(\xi, y)$. Then:*

(i) *After suitable normalization of the N -th order truncated Evans function D_N , the limit*

$$D = \lim_{N \rightarrow \infty} D_N$$

exists on $\overline{\mathbb{C}_+}$. This D satisfies

$$D(0) = 0 \quad \text{and} \quad D'(0) = 0.$$

(ii) *If the amplitude of U^c is sufficiently small, D satisfies, in addition,*

$$D''(0) \neq 0, \quad \text{and} \quad D(\kappa) \neq 0 \text{ for } \kappa \in \overline{\mathbb{C}_+} \setminus \{0\}.$$

A natural next step towards proving part (ii) of this conjecture consists of investigating whether the result on small-amplitude waves in Sec. 4 allows for taking the limit $N \rightarrow \infty$. More concretely, we ask at first whether the two functions $\zeta^+(\xi; \varepsilon, N)$ and $\eta^-(\xi; \varepsilon, N)$, which span the stable space and the dual of the unstable space, respectively, converge in an appropriate sense as $N \rightarrow \infty$. This is obvious for $\varepsilon = 0$ because $\zeta^+(\xi; \varepsilon, N+1)$ is obtained from $\zeta^+(\xi; \varepsilon, N)$ simply by appending a zero (and similarly for η^-). Would such a result be true for $\varepsilon > 0$ as well?

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